Quasiperiodic waves and asymptotic behavior for Bogoyavlenskii's breaking soliton equation in (2+1) dimensions

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Based on a multidimensional Riemann theta function, the Hirota bilinear method is extended to explicitly construct multiperiodic (quasiperiodic) wave solutions for the (2+1)-dimensional Bogoyavlenskii breaking soliton equation. Among these periodic waves, the one-periodic waves are well-known cnoidal waves, their surface pattern is one-dimensional, and often they are used as one-dimensional models of periodic waves in shallow water. The two-periodic (biperiodic) waves are a direct generalization of one-periodic waves, their surface pattern is two dimensional, that is, they have two independent spatial periods in two independent horizontal directions. The two-periodic waves may be considered to represent periodic waves in shallow water without the assumption of one dimensionality. A limiting procedure is presented to analyze asymptotic behavior of the one- and two-periodic waves in details. The exact relations between the periodic wave solutions and the well-known soliton solutions are established. It is rigorously shown that the periodic wave solutions tend to the soliton solutions under a "small amplitude" limit.

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I. INTRODUCTION

In recent years, there has been much interest in investigating different kinds of exact solutions of nonlinear evolution equations, such as soliton, negaton, peakon, complexiton, cuspon, rational, periodic, and quasiperiodic solutions [1–32]. Exact solutions play an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma, and elastic media are often modeled by the bell-shaped sech solutions and the kink shaped tanh solutions. The exact solutions, if available, of those nonlinear equations can facilitate the verification of numerical solvers and aid in the stability analysis of solutions. However, investigating or establishing relations among different exact solutions is also a very interesting topic. Since these relations not only provide an approach to deforming exact solutions, but also help us to study the structures and properties of some complicated forms of the solutions such as quasiperiodic solutions.

The quasiperiodic solutions (also called algebrogeometric solutions or finite gap solutions) of nonlinear equations were originally obtained on the Korteweg–de Vries (KdV) equation based inverse spectral theory and algebro-geometric method developed by pioneers such as Novikov, Dubrovin, Mckean, Lax, Its, Matveev, and co-workers [4–8] in the late 1970's. By now this theory has been extended to a large class of nonlinear integrable equations including the sine-Gordon equation, Camassa-Holm equation, Thirring model equation, Kadomtsev-Petviashvili equation, Ablowitz-Ladik lattice, and Toda lattice [9–19]. The quasiperiodic solutions describe the nonlinear interaction of several modes. All the main physical characteristics of the quasiperiodic solutions (wave numbers, phase velocities, amplitudes of the interacting modes) are defined by a compact Riemann surface. There are numerous applications of the finite-gap integration theory in condensed matter physics, state physics, and fluid mechanics. For example, in the Peierls state, phonon produce a finite-gap potential for electrons, and the Peierls state is a lattice of solutions at low densities of electrons [9]. A most famous mechanical system, the Kowalewski top, was the focus of interest in the 19th century. The equation of motion of the top can be solved through finite-gap theory [9]. A problem of fundamental interest in fluid mechanics is to provide an accurate description of waves on a water surface. The Kadomtsev-Petviashvili (KP) equation is known to describe the evolution of waves in shallow water. The KP equation admits a large family of quasiperiodic solutions. Each solution has N independent phases. Experiments demonstrate the existence of genuinely two-dimensional shallow water waves that are full periodic in two spatial directions and time. The comparisons with experiments showed that the two-periodic wave solutions of the KP equation describe shallow water waves with much accuracy [33,34].

However, using the finite-gap (algebrogeometric) theory is rather difficult to directly determine the characteristic parameters of waves such as frequencies and phase shifts for a function of given wave numbers and amplitudes. In 1980s, Nakamura proposed a convenient way to construct a kind of quasiperiodic solutions of nonlinear equations in his two serial papers [35,36], where the periodic wave solutions of the KdV equation and the Boussinesq equation were obtained by means of the Hirota's bilinear method [37–39]. This method not only conveniently obtains periodic solutions of a nonlinear equation, but also directly gives the explicit relations among frequencies, wave numbers, phase shifts, and ampli-

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tudes of the wave. Recently, we have extended this method to investigate the discrete Toda lattice [40]. But the asymptotic properties for this type of periodic wave solutions still have not been considered in detail [35,36,40,41].

One objective of this paper is to provide a detail asymptotic analysis procedure to this kind of periodic waves by considering the following (2+1)-dimensional Bogoyavlenskii's breaking soliton equation as an illustrative example

$$u_t + u_{xxy} + 4uu_y + 4u_x\partial^{-1}u_y = 0, \tag{1.1}$$

its equivalent form

$$u_t + u_{xxy} + 4uv_x + 4u_xv = 0,$$

$$u_y = v_x,$$

which describes the (2+1)-dimensional interaction of a Riemann wave propagating along the y axis with a long wave along the x axis. The $u=u(x,y,t)$ represents the physical field and $v=v(x,y,t)$ some potential. This equation is typical of the so-called "breaking soliton" equation and was studied by Bogoyovenskii, where overlapping solutions were generated [42]. Radha and Lakshmanan showed that Eq. (1.1) possesses the Painlevé property and dromionlike structures [43]. Ikeda and Takasaki presented a Bogoyovenskii's hierarchy and its breaking soliton solutions [44]. In recent years, many papers have been focusing their topics on various exact solutions of Eq. (1.1) including soliton solutions, and Jacobi or Weierstrass elliptic periodic solutions [45–50]. However, these periodic wave solutions are actually one-dimensional cnoidal waves (one-dimensional surface patterns). One of the major shortcomings of cnoidal theory as a practical model of water waves is that the theory is one dimensional, whereas the water surface is two dimensional. It follows that cnoidal waves are necessarily long-crested, whereas both long-crested and shorted waves are observed in shallow water.

So another objective of this paper is to provide a multidimensional generalization of cnoidal waves for Eq. (1.1). The organization of this paper is as follows. In Sec. II, we briefly introduce a useful bilinear form of Eq. (1.1), the Riemann theta function, and its periodicity. In Secs. III and IV, we apply Hirota's bilinear method to construct one- and two-periodic wave solutions of Eq. (1.1), respectively. The one-periodic waves are well-known cnoidal waves, and their surface pattern is one-dimensional. The two-periodic waves, whose surface pattern is two dimensional, are a direct generalization of one-periodic waves. We further apply a limiting procedure to analyze the features and asymptotic behavior of the one- and two-periodic wave solutions in detail. It is rigorously shown that the periodic solutions tend to the known soliton solutions under a "small amplitude" limit.

II. THE BILINEAR FORM AND THE RIEMANN THETA FUNCTION

In this section, we briefly introduce a useful bilinear form of Eq. (1.1) and some main points on the Riemann theta function.

A. The bilinear form of Eq. (1.1)

The Hirota bilinear method is powerful in constructing exact solutions for a large number of nonlinear equations. Once a nonlinear equation is written in bilinear forms by a dependent variable transformation, then multisoliton solutions are usually obtained.

By the dependent variable transformation [43,49]

$$u = \frac{3}{2}\partial_x^2 \ln f(x,y,t),$$

Eq. (1.1) is then transformed into a bilinear form

$$(D_tD_x + D_yD_x^3)f(x,y,t)f(x,y,t) = 0, \tag{2.1}$$

where the bilinear differential operators D_x, D_y, and D_t are defined by

$$D_x^mD_y^nD_t^kf(x,y,t)g(x,y,t) = (\partial_x - \partial_{x'})^m(\partial_y - \partial_{y'})^n(\partial_t - \partial_{t'})^k \times f(x,y,t)g(x',y',t')|_{x'=x,y'=y,t'=t}.$$

These operators have a good property when acting on exponential functions, namely,

$$D_x^mD_y^nD_t^ke^{\xi_1}e^{\xi_2} = (\alpha_1 - \alpha_2)^m(\rho_1 - \rho_2)^n(\omega_1 - \omega_2)^ke^{\xi_1+\xi_2},$$

where $\xi_j=\alpha_jx+\rho_jy+\omega_jt+\delta_j$, $j=1,2$. More generally, we have

$$G(D_x,D_y,D_t)e^{\xi_1}e^{\xi_2} = G(\alpha_1 - \alpha_2,\rho_1 - \rho_2,\omega_1 - \omega_2)e^{\xi_1+\xi_2}, \tag{2.2}$$

where $G(D_x,D_y,D_t)$ is a polynomial about D_x, D_y, and D_t. This property will be used later and plays a key role in the construction of the periodic wave solutions.

Following the Hirota bilinear theory, Eq. (1.1) admits a one-soliton solution

$$u_1 = \frac{3}{2}\partial_x^2 \ln(1 + e^{\eta}), \tag{2.3}$$

with phase variable $\eta=\mu x+\nu y-\mu^2\nu t+\gamma$, and μ, ν, γ being constants. While the two-soliton solution takes the form

$$u_2 = \frac{3}{2}\partial_x^2 \ln(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}), \tag{2.4}$$

with

$$\eta_j = \mu_jx + \nu_jy - \mu_j^2\nu_jt + \gamma_j, \quad j = 1,2,$$

$$e^{A_{12}} = \frac{(\mu_1 - \mu_2)[\mu_2\nu_1(2\mu_1 - \mu_2) - \mu_1\nu_2(2\mu_2 - \mu_1)]}{(\mu_1 + \mu_2)[\mu_2\nu_1(2\mu_1 + \mu_2) - \mu_1\nu_2(2\mu_2 + \mu_1)]},$$

and here μ_j, ν_j, γ_j, $j=1,2$ are free constants.

To apply the Hirota bilinear method for constructing multi-periodic wave solutions of the Eq. (1.1), we consider a slightly generalized form of the bilinear equation (2.1). Here we assume Eq. (1.1) with the nonzero asymptotic condition $u \to u_0$ as $|\xi| \to 0$, and look for its solution in the form

$$u = u_0 + \frac{3}{2}\partial_x^2 \ln \vartheta(\xi), \qquad (2.5)$$

where u_0 is a constant solution of Eq. (1.1), and phase variable ξ is taken as the form $\xi=(\xi_1,\ldots,\xi_N)^T$, $\xi_j=\alpha_j x+\rho_j y+\omega_j t+\delta_j$, $j=1,2,\ldots,N$.

By substituting Eq. (2.5) into Eq. (1.1) and integrating with respect to x, we then get the following bilinear form:

$$G(D_x,D_y,D_t)\vartheta(\xi)\vartheta(\xi) = (D_tD_x + D_yD_x^3 + u_0D_yD_x + c)\vartheta(\xi)\vartheta(\xi) = 0, \quad (2.6)$$

where $c=c(y,t)$ is an integration constant. For the bilinear equation (2.6), we are interested in its multiperiodic solutions in terms of the Riemann theta function $\vartheta(\xi)$.

Remark 1. The constant $c=c(y,t)$ may be taken to be zero in the construction of soliton solutions. But in our present periodic case, the nonzero constant c plays an important role and must not be dropped. Because elliptic functions generally do not satisfy equations with zero integration constants such as Eq. (2.1). For example, consider the well-known KdV equation

$$4u_t = 6uu_x + u_{xxx}.$$

By setting $u=u(\xi)=u(x-\nu t)$ and after an elementary integration, we have

$$-4\nu u'^2 = 2u^3 + 2c_1u + c_2,$$

where c_1 and c_2 are two integration constants. For the zero integration constants, we get a well-known soliton solution. While for nonzero integration constants, we obtain a periodic solution

$$u = -2\wp(\xi) + \gamma,$$

where both the Weierstrass elliptic function $\wp(\xi)$ and γ depend on the integration constants c_1 and c_2. ■

B. The theta function and its periodicity

Let us consider multiperiodic wave solutions of Eq. (1.1) based on the following multidimensional Riemann theta function of genus N

$$\vartheta(\xi) = \vartheta(\xi,\tau) = \sum_{n\in Z^N} e^{-\pi\langle \tau n,n\rangle + 2\pi i\langle \xi,n\rangle}. \qquad (2.7)$$

Here the integer value vector $n=(n_1,\ldots,n_N)^T \in Z^N$, and complex phase variables $\xi=(\xi_1,\ldots,\xi_N)^T \in C^N$. Moreover, for two vectors $f=(f_1,\ldots,f_N)^T$ and $g=(g_1,\ldots,g_N)^T$, their inner product is defined by

$$\langle f,g\rangle = f_1g_1 + f_2g_2 + \cdots + f_Ng_N.$$

The $\tau=(\tau_{ij})$ is a positive definite and real-valued symmetric $N\times N$ matrix, which we call the period matrix of the theta function. The entries τ_{ij} of the period matrix τ can be considered as free parameters of the theta function (2.7). Under these conditions, the Fourier series (2.7) converges to a real-valued function for an arbitrary vector $\xi \in C^N$.

Remark 2. In the construction of periodic wave solution by using an algebrogeometric method [4–19], The matrix τ is usually constructed via a compact Riemann surface Γ of genus $N \in \mathbb{N}$. We take two sets of regular cycle paths $a_1,a_2,\ldots,a_N$; $b_1,b_2,\ldots,b_N$ on Γ in such a way that the intersection numbers of cycles satisfy

$$a_k \circ a_j = b_k \circ b_j = 0, a_k \circ b_j = \delta_{kj}, \quad k,j = 1,\ldots,N.$$

We choose the normalized holomorphic differentials $\omega_j, j=1,\ldots,N$ on Γ and let

$$a_{jk} = \int_{a_k} \omega_j, \quad b_{jk} = \int_{b_k} \omega_j,$$

then $N\times N$ matrices $A=(b_{jk})$ and $B=(b_{jk})$ are invertible. Define matrices C and τ by

$$C = (c_{jk}) = A^{-1}, \quad \tau = (\tau_{jk}) = A^{-1}B.$$

It is can be shown that the matrix τ is symmetric and has positive definite imaginary part. In this paper, we take the τ to be pure imaginary matrix to make the theta function (2.7) real valued. ■

In the following, we discuss the quasiperiodicity of the theta function $\vartheta(\xi,\tau)$, which plays a central role in this paper.

Definition 1. A function $g(x,t)$ on $\mathbb{C}^N \times \mathbb{C}$ is said to be quasiperiodic in t with fundamental periods $T_1,\ldots,T_k \in \mathbb{C}$ if $T_1,\ldots,T_k$ are linearly dependent over $\mathbb{Z}$ and there exist a function $G(x,t) \in \mathbb{C}^N \times \mathbb{C}^k$ such that for $\forall (y_1,\ldots,y_k) \in \mathbb{C}^k$,

$$G(x,y_1,\ldots,y_{j-1},y_j+T_j,y_{j+1},\ldots,y_k) = G(x,y_1,\ldots,y_{j-1},y_j,y_{j+1},\ldots,y_k),$$

$$G(x,t,\ldots,t,\ldots,t) = g(x,t).$$

In particular, $g(x,t)$ becomes periodic with T if and only if $T_j=m_jT$. ■

Example 1. The function $g(x,t)=a(x)\cos(t)+b(x)\cos(2t)+c(x)\cos(\sqrt{3}t)$ is quasiperiodic in t, with

$$G(x,y_1,y_2,y_3) = a(x)\cos(y_1) + b(x)\cos(2y_2) + c(x)\cos(\sqrt{3}y_3),$$

$$T_1 = 2\pi, \quad T_2 = \pi, \quad T_3 = 2\pi/\sqrt{3}.$$

Let us see the periodicity of the theta function $\vartheta(\xi)$.

Proposition 1. [51]. Let e_j be the jth column of $N\times N$ identity matrix I_N, τ_j the jth column of τ, and τ_{jj} the (j,j) entry of τ. Then the theta function $\vartheta(\xi)$ has the periodic properties

$$\vartheta(\xi + e_j,\tau) = \vartheta(\xi,\tau), \qquad (2.8)$$

$$\vartheta(\xi + i\tau_j, \tau) = e^{-2\pi i \xi_j + \pi\tau_{jj}} \vartheta(\xi, \tau). \qquad (2.9)$$

If considering the $N\times 2N$ matrixes $\Omega=(I_N, i\tau)$ and $\Lambda=(0,-I_N)$. We can unify Eqs. (2.8) and (2.9) as a general form

$$\vartheta(\xi + \Omega\bar{e}_j, \tau) = e^{2\pi i\langle \Lambda \bar{e}_j, \xi\rangle + \varsigma_j} \vartheta(\xi, \tau), \qquad (2.10)$$

where $\bar{e}_j$, $j=1,\ldots,2N$ denote the $2Nth$ columns of the $2N\times 2N$ identity matrix, and ς_j is the jth component of the $2N$-dimensional vector $\varsigma=(0,\ldots,0,\pi\tau_{11},\ldots,\pi\tau_{NN})^T$.

The theta function $\vartheta(\xi)$ which satisfies the condition (2.10) is called a multiplicative function. We regard the vectors $\{e_j, j=1,\ldots,N\}$ and $\{i\tau_j, j=1,\ldots,N\}$ as periods of the theta function $\vartheta(\xi)$ with multipliers 1 and $e^{-2\pi i\xi_j+\pi\tau_{jj}}$, respectively. Of course, only the first N vectors are actually periods of the theta function $\vartheta(\xi,\tau)$, but the last N vectors are the periods of the function $\partial^2_{\xi_j} \ln \vartheta(\xi,\tau)$, $j=1,\ldots,N$.

Proposition 2. Let e_j and τ_j be defined as above proposition 1, then we have

$$\partial^2_{\xi_j} \ln \vartheta(\xi + e_j, \tau) = \partial^2_{\xi_j} \ln \vartheta(\xi, \tau), \quad j=1,\ldots,N, \qquad (2.11)$$

$$\partial^2_{\xi_j} \ln \vartheta(\xi + i\tau_j, \tau) = \partial^2_{\xi_j} \ln \vartheta(\xi, \tau), \quad j=1,\ldots,N. \qquad (2.12)$$

Proof. Equation (2.11) is clear from Eq. (2.8). By using Eq. (2.9), it is easy to see that

$$\frac{\vartheta'_{\xi_j}(\xi + i\tau_j, \tau)}{\vartheta(\xi + i\tau_j, \tau)} = -2\pi i + \frac{\vartheta'_{\xi_j}(\xi, \tau)}{\vartheta(\xi, \tau)},$$

that is,

$$\partial_{\xi_j} \ln \vartheta(\xi + i\tau_j, \tau) = -2\pi i + \partial_{\xi_j} \ln \vartheta(\xi, \tau). \qquad (2.13)$$

Differentiating Eq. (2.13) with respective to ξ_j again immediately leads to Eq. (2.12). Equations (2.11) and (2.12) indicate that for each $j=1,\ldots,N$, the function $\partial^2_{\xi_j} \ln \vartheta(\xi,\tau)$ is periodic with two fundamental periods e_j and $i\tau_j$. ■

C. The periodicity of solution (2.5)

Now we turn to see the periodicity of the solution (2.5). According to the differential relation $\partial_x^2 = \alpha_j^2 \partial^2_{\xi_j}$, $j=1,\ldots,N$ and proposition 2, the solution (2.5) is a quasiperiodic function with $2N$ fundamental periods $\{e_j,\ldots,e_N\}$ and $\{i\tau_j,\ldots,i\tau_N\}$. The "quasiperiodic" means that u is periodic in each of the N phases $\{\xi_j,\ldots,\xi_N\}$, if the other $N-1$ phases are fixed.

In the simplest case when $N=1$, the solution (2.5) reproduces the cnoidal waves, which can be expressed as the Weierstrass or Jacobi elliptic form according to the following relations:

$$\wp(\xi,\tau) = -[\ln \vartheta_{11}(\xi,\tau)'' + c],$$

$$\mathrm{cn}[\pi\vartheta(0,\tau)\xi, k] = \frac{\vartheta_{01}(0,\tau)\vartheta_{10}(\xi,\tau)}{\vartheta_{10}(0,\tau)\vartheta_{01}(\xi,\tau)}, \quad k = \left(\frac{\vartheta_{10}(0,\tau)}{\vartheta(0,\tau)}\right)^2,$$

where c is defined so that the Laurent expansion of $\wp(\xi,\tau)$ at $\xi=0$ has zero constant term; k is called the modulus of the Jacobi elliptic function. Three auxiliary (or half-period) theta functions are defined by

$$\vartheta_{01}(\xi,\tau) = \vartheta\left(\xi + \frac{1}{2}, \tau\right),$$

$$\vartheta_{10}(\xi,\tau) = e^{-1/4\pi\tau + i\pi\xi}\vartheta\left(\xi + i\frac{1}{2}\tau, \tau\right),$$

$$\vartheta_{11}(\xi,\tau) = e^{-1/4\pi\tau + i\pi(\xi+1/2)}\vartheta\left(\xi + i\frac{1}{2}\tau + \frac{1}{2}, \tau\right).$$

In mathematical physics, the Weierstrass and Jacobi elliptic functions are two basic elliptic functions, while the theta function is a special function in complex variables. They are important in several areas, including the theories of Abelian varieties and moduli spaces. They have also been applied to soliton, quantum field, and specifically string theory. The details about these functions refer, for instance, to standard monographs [51–53]. The waves of interest in this paper appear at the case when $N=2$, then the solution (2.5) is periodic in two independent horizontal directions.

III. ONE-PERIODIC WAVES AND ASYMPTOTIC PROPERTIES

In this section, we consider one-periodic wave solutions of Eq. (1.1). We first consider the simple case when $N=1$, then theta function (2.7) reduces the following Fourier series in n:

$$\vartheta(\xi,\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n\xi - \pi n^2\tau}, \qquad (3.1)$$

where the phase variable $\xi=\alpha x+\rho y+\omega t+\delta$, and the parameter $\tau>0$.

A. Construction of one-periodic waves

To make the theta function (3.1) be a solution of the bilinear equation (2.6), we substitute Eq. (3.1) into the left side of Eq. (2.6) and by using the property (2.2) obtain that

$$G(D_x,D_y,D_t)\vartheta(\xi,\tau)\cdot\vartheta(\xi,\tau)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G(D_x,D_y,D_t)e^{2\pi in\xi-\pi n^2\tau}e^{2\pi im\xi-\pi m^2\tau}$$
$$=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G[2\pi i(n-m)\alpha,2\pi i(n-m)\rho,2\pi i(n-m)\omega]e^{2\pi i(n+m)\xi-\pi(n^2+m^2)\tau}$$
$$\overset{m=m'-n}{=}\sum_{m'=-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty}G(2\pi i(2n-m')\alpha,2\pi i(2n-m')\rho,2\pi i(2n-m')\omega)e^{-\pi[(n^2+(n-m')^2]\tau}\right\}e^{2\pi im'\xi}$$
$$=\sum_{m'=-\infty}^{\infty}\bar{G}(m')e^{2\pi im'\xi},$$

where in the last line we have denoted the coefficient of $e^{2\pi im'\xi}$ in the above equation as

$$\bar{G}(m')=\sum_{n=-\infty}^{\infty}G[2\pi i(2n-m')\alpha,2\pi i(2n-m')\rho,2\pi i(2n-m')\omega]e^{-\pi[n^2+(n-m')^2]\tau}. \quad (3.2)$$

In the following, we compute each series $\bar{G}(m')$ for $m'\in Z$. By shifting summation index by $n=n'+1$, we have the following fact:

$$\bar{G}(m')=\left(\sum_{n'=-\infty}^{\infty}G\{2\pi i[2n'-(m'-2)]\alpha,2\pi i[2n'-(m'-2)]\rho,2\pi i[2n'-(m'-2)]\omega\}\right.$$
$$\times\exp(-\pi\{[n'^2+(n'-(m'-2)\ ]^2\}\tau)\exp[-2\pi(m'-1)\tau]\ =\bar{G}(m'-2)e^{-2\pi(m'-1)\tau}=\cdots$$
$$=\begin{cases}\bar{G}(0)e^{-\pi m'^2\tau/2}, & m' \text{ is even},\\ \bar{G}(1)e^{-\pi(m'^2-1)\tau/4}, & m' \text{ is odd},\end{cases}$$

which implies that $\bar{G}(m'),m'\in Z$ are completely dominated by two function $\bar{G}(0)$ and $\bar{G}(1)$. In other word, if the following two equations are satisfied:

$$\bar{G}(0)=\bar{G}(1)=0, \quad (3.3)$$

then it follows that

$$\bar{G}(m')=0,\quad m'\in Z$$

and thus the theta function (3.1) is an exact solution to Eq. (2.6), namely,

$$G(D_x,D_y,D_t)\vartheta(\xi)\vartheta(\xi)=0.$$

It follows from Eqs. (3.2) and (3.3) that

$$\bar{G}(0)=\sum_{n=-\infty}^{\infty}(-16\pi^2n^2\alpha\omega+256\pi^4n^4\alpha^3\rho$$
$$-16u_0\pi^2n^2\alpha\rho+c)e^{-2\pi n^2\tau}=0,$$

$$\bar{G}(1)=\sum_{n=-\infty}^{\infty}[-4\pi^2(2n-1)^2\alpha\omega+16\pi^4(2n-1)^4\alpha^3\rho$$
$$-4u_0\pi^2(2n-1)^2\alpha\rho+c]e^{-\pi(2n^2-2n+1)\tau}=0. \quad (3.4)$$

By introducing the notations as

$$\lambda=e^{-\pi\tau},\quad a_{11}=-\sum_{n=-\infty}^{\infty}16\pi^2n^2\alpha\lambda^{2n^2},$$

$$a_{12}=\sum_{n=-\infty}^{\infty}\lambda^{2n^2},\quad a_{22}=\sum_{n=-\infty}^{\infty}\lambda^{2n^2-2n+1},$$

$$a_{21}=-\sum_{n=-\infty}^{\infty}4\pi^2(2n-1)^2\alpha\lambda^{2n^2-2n+1},$$

$$b_1=-\sum_{n=-\infty}^{\infty}(256\pi^4n^4\alpha^3\rho-16u_0\pi^2n^2\alpha\rho)\lambda^{2n^2},$$

$$b_2 = -\sum_{n=-\infty}^{\infty} [16\pi^4(2n-1)^4\alpha^2 - 4u_0\pi^2(2n-1)^2]\alpha\rho\lambda^{2n^2-2n+1}, \tag{3.5}$$

we simply change Eq. (3.4) into a linear system about the frequency ω and the integration constant c, namely,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \omega \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{3.6}$$

Now we get a one-periodic wave solution of Eq. (1.1)

$$u = u_0 + \frac{3}{2}\partial_x^2 \ln \vartheta(\xi), \tag{3.7}$$

provided the vector $(\omega, c)^T$ solves Eq. (3.6) with the theta function $\vartheta(\xi)$ given by Eq. (3.1) and parameters ω, c by Eq. (3.6). The other parameters α, ρ, τ, δ, and u_0 are free. The three parameters α, ρ, and τ completely dominate a one-periodic wave. Figure 1 shows a one-periodic wave for one choice of the parameters.

B. Feature and asymptotic property of one-periodic waves

In summary, the one-periodic wave (3.7) has a simple characterization. (i) It is real valued and bounded for all complex variables (x,y,t). (ii) It is actually a kind of one-dimensional cnoidal waves, i.e., there is a single phase variable ξ. Its speed parameter is given by

$$\omega = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}.$$

Their surface pattern is one-dimensional and they are often used as one-dimensional models of periodic waves in shallow water. (iii) It has two fundamental periods 1 and $i\tau$ in the phase variable ξ. (iv) It has only one wave pattern for all time, and it can be viewed as a parallel superposition of overlapping one-solitary waves, placed one period apart (see Fig. 1).

In the following, we further consider asymptotic properties of the one-periodic wave solution. For this purpose, we have to use the solutions of the system (3.6). Since both the coefficient matrix and the right-side vector of system (3.6) are power series about λ, its solution $(\omega, c)^T$ also should be a series about λ. We can solve system (3.6) via small parameter expansion method and general procedure is described as follows.

We write the coefficient matrix and the right-side vector of system (3.6) into power series of λ

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A_0 + A_1\lambda + A_2\lambda^2 + \cdots \tag{3.8}$$

and

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = B_0 + B_1\lambda + B_2\lambda^2 + \cdots . \tag{3.9}$$

Again suppose that the solution of system (3.6) is in the form

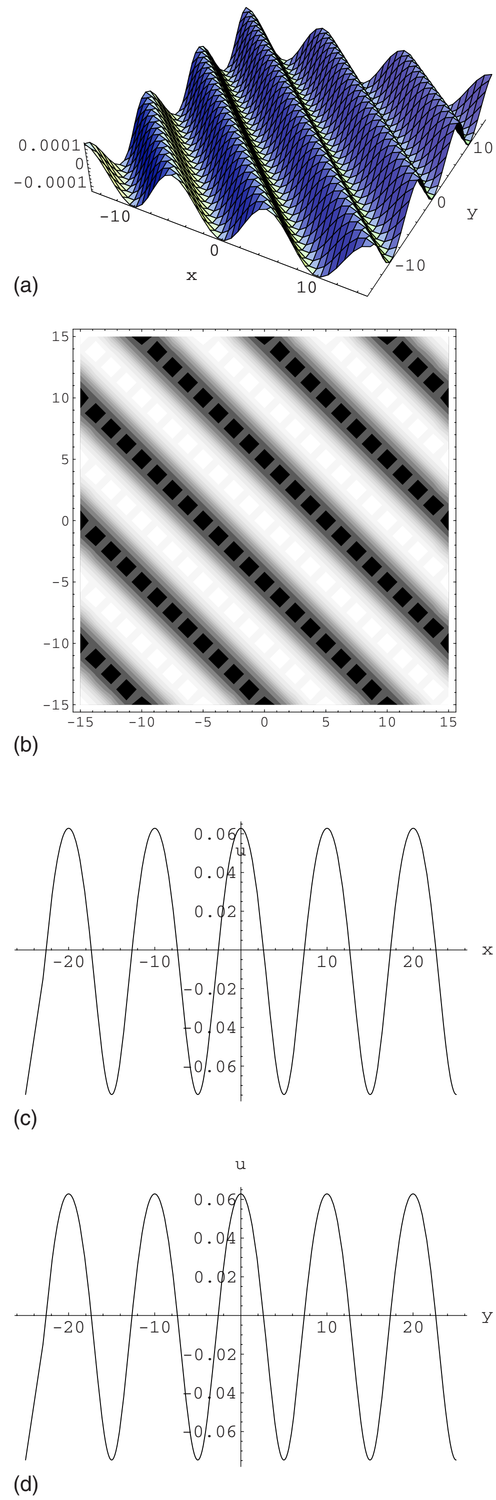

FIG. 1. (Color online) A one-periodic wave of the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation with parameters: $u_0=\delta=0$, $\alpha=0.1$, $\tau=3$, $\rho=1$. This figure shows that every one-periodic wave is one dimensional, and it can be viewed as a superposition of overlapping solitary waves, placed one period apart. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs. (c) Wave propagation pattern of the wave along the x axis. (d) Wave propagation pattern of the wave along the y axis.

$$\begin{pmatrix} \omega \\ c \end{pmatrix} = X_0 + X_1\lambda + X_2\lambda^2 + \cdots . \tag{3.10}$$

Substituting Eqs. (3.8)–(3.10) into Eq. (3.6) leads to the following recursion relations:

$$A_0X_0 = B_0,$$

$$A_0X_1 + A_1X_0 = B_1,$$

$$A_0X_2 + A_2X_0 + A_1X_1 = B_2,$$

$$\cdots,$$

$$A_0X_k + A_1X_{k-1} + \cdots + A_kX_0 = B_k, \tag{3.11}$$

form which we then recursively get each vector X_j, $j = 0,1,\ldots$.

If the matrix A_0 is reversible, solving Eq. (3.11) gives

$$X_0 = A_0^{-1}B_0,$$

$$X_k = A_0^{-1}\left(B_k - \sum_{j=1}^{k} A_jB_{k-1}\right), \quad k = 1,2,\ldots .$$

If A_0 and A_1 are not inverse, but they take the following form (which will be used in the proof of the following theorem 1):

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -8\pi^2\alpha & 2 \end{pmatrix},$$

solving relations (3.11) gives

$$X_0 = \begin{pmatrix} -\dfrac{1}{8\pi^2\alpha}(B_1^{(\mathrm{II})} - 2B_0^{(\mathrm{I})}) \\ B_0^{(\mathrm{I})} \end{pmatrix},$$

$$X_1 = \begin{pmatrix} -\dfrac{1}{8\pi^2\alpha}[(B_2 - A_2X_0)^{(\mathrm{II})} - 2B_1^{(\mathrm{I})}] \\ B_1^{(\mathrm{I})} \end{pmatrix},$$

$$X_k = \begin{pmatrix} -\dfrac{1}{8\pi^2\alpha}\left[\left(B_{k+1} - \displaystyle\sum_{j=2}^{k+1} A_jX_{k+1-j}\right)^{(\mathrm{II})} - 2\left(B_{k+1} - \displaystyle\sum_{j=2}^{k} A_jX_{k-j}\right)^{(\mathrm{I})}\right] \\ \left(B_{k+1} - \displaystyle\sum_{j=2}^{k} A_jX_{k-j}\right)^{(\mathrm{I})} \end{pmatrix}, \quad k = 2,3,\ldots , \tag{3.12}$$

where $V^{(\mathrm{I})}$ and $V^{(\mathrm{II})}$ denote the first and second component of a two-dimensional vector V, respectively. Interestingly, the relation between the one-periodic wave solution (3.7) and the one-soliton solution (2.3) can be established as follows.

Theorem 1. Suppose that the vector $(\omega,c)^T$ is a solution of the system (3.6), and for the one-periodic wave solution (3.7), we let

$$u_0 = 0, \quad \alpha = \frac{\mu}{2\pi i}, \quad \rho = \frac{\nu}{2\pi i}, \quad \delta = \frac{\gamma + \pi\tau}{2\pi i}, \tag{3.13}$$

where μ, ν, and γ are the same as those in Eq. (2.3). Then we have the following asymptotic properties:

$$c \to 0 \quad ,\xi \to \frac{\eta + \pi\tau}{2\pi i}, \quad \vartheta(\xi,\tau) \to 1 + e^{\eta},$$

$$\text{as } \lambda \to 0. \tag{3.14}$$

In other words, the one-periodic solution (3.7) tends to the one-soliton solution (2.3) under a small amplitude limit, that is,

$$u \to u_1, \quad \text{as } \lambda \to 0.$$

Proof. By using Eq. (3.5), we write functions a_{ij}, b_j, i, $j = 1,2$ as the series about λ

$$a_{11} = -32\pi^2\alpha(\lambda^2 + 4\lambda^8 + 9\lambda^{18} + \cdots),$$

$$a_{12} = 1 + 2\lambda^2 + 2\lambda^8 + 2\lambda^{18} + 2\lambda^{32} + \cdots,$$

$$a_{21} = -8\pi^2\alpha(\lambda + 9\lambda^5 + 25\lambda^{13} + \cdots),$$

$$a_{22} = 2\lambda + 2\lambda^5 + 2\lambda^{13} + 2\lambda^{25} + \cdots,$$

$$b_1 = -512\pi^4\alpha^3\rho(\lambda^2 + 8\lambda^8 + \cdots),$$

$$b_2 = -32\pi^4\alpha^3\rho(\lambda + 81\lambda^5 + 625\lambda^{13} + \cdots). \tag{3.15}$$

Thus, using Eqs. (3.8) and (3.9), we have

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -8\pi^2\alpha & 2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -32\pi^2\alpha & 2 \\ 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 \\ -72\pi^2\alpha & 2 \end{pmatrix},$$

$$A_3 = A_4 = 0,$$

$$\cdots, \tag{3.16}$$

and

$$B_0=0, \quad B_1=\begin{pmatrix} 0 \\ -32\pi^4\alpha^3\rho \end{pmatrix},$$

$$B_2=\begin{pmatrix} -512\pi^4\alpha^3\rho \\ 0 \end{pmatrix}, \quad B_5=\begin{pmatrix} 0 \\ -2592\pi^2\alpha^3\rho \end{pmatrix},$$

$$B_3=B_4=0,$$

$$\cdots . \tag{3.17}$$

Substituting Eqs. (3.16) and (3.17) into formulas (3.12), we then obtain

$$X_0=\begin{pmatrix} 4\pi^2\alpha^2\rho \\ 0 \end{pmatrix}, \quad X_1=X_3=0, \quad X_2=\begin{pmatrix} 384\pi^4\alpha^3\rho \\ -384\pi^4\alpha^3\rho \end{pmatrix},$$

$$X_4=\begin{pmatrix} -2304\pi^4\alpha^3\rho \\ 384\pi^4\alpha^3\rho(32\pi^2\alpha+2) \end{pmatrix}, \ldots ,$$

and thus

$$c=-384\pi^4\alpha^3\rho\lambda^2+384\pi^4\alpha^3\rho(32\pi^2\alpha+2)\lambda^4+o(\lambda^4),$$

$$\omega=4\pi^2\alpha^2\rho+384\pi^4\alpha^3\rho\lambda^2-2304\pi^4\alpha^3\rho\lambda^4+o(\lambda^4),$$

which exactly implies by using relation (3.13) that

$$c\to 0, \quad 2\pi i\omega \to 8\pi^3 i\alpha^2\rho=-\mu^2\nu, \quad \text{as } \lambda\to 0. \tag{3.18}$$

It remains to show that the one-periodic wave (3.7) degenerates to the one-soliton solution (2.3) under the limit $\lambda \to 0$. For this purpose, we first expand the periodic function $\vartheta(\xi)$ in the form

$$\vartheta(\xi,\tau)=1+\lambda(e^{2\pi i\xi}+e^{-2\pi i\xi})+\lambda^4(e^{4\pi i\xi}+e^{-4\pi i\xi})+\cdots .$$

By using the transformation (3.13), it follows that

$$\vartheta(\xi,\tau)=1+e^{\hat{\xi}}+\lambda^2(e^{-\hat{\xi}}+e^{2\hat{\xi}})+\lambda^6(e^{-2\hat{\xi}}+e^{3\hat{\xi}})+\cdots$$

$$\to 1+e^{\hat{\xi}}, \quad \text{as } \lambda\to 0, \tag{3.19}$$

where

$$\hat{\xi}=2\pi i\xi-\pi\tau=\mu x+\nu y+2\pi i\omega t+\gamma. \tag{3.20}$$

Combining Eqs. (3.18) and (3.20) deduces that

$$\hat{\xi}\to \mu x+\nu y-\mu^2\nu+\gamma=\eta, \quad \text{as } \lambda\to 0 \tag{3.21}$$

or, equivalently,

$$\xi\to\frac{\eta+\pi\tau}{2\pi i}, \quad \text{as } \lambda\to 0.$$

Again Eqs. (3.19) and (3.21) immediately leads to

$$\vartheta(\xi,\tau)\to 1+e^{\eta}, \quad \text{as } \lambda\to 0.$$

Therefore we conclude that the one-periodic solution (3.7) just goes to the one-soliton solution (2.3) as the amplitude $\lambda\to 0$. ■

IV. TWO-PERIODIC WAVES AND ASYMPTOTIC PROPERTIES

In this section, we consider two-periodic wave solutions to Eq. (1.1), which are a two-dimensional generalization of one-periodic wave solutions. The two-periodic waves of interest here have three-dimensional velocity fields and two-dimensional surface patterns.

A. Construction of two-periodic waves

In the case when $N=2$, the Riemann theta function (2.7) takes the form

$$\vartheta(\xi,\tau)=\vartheta(\xi_1,\xi_2,\tau)=\sum_{n\in Z^2} e^{2\pi i\langle\xi,n\rangle-\pi\langle\tau n,n\rangle}, \tag{4.1}$$

where $n=(n_1,n_2)^T\in Z^2$, $\xi=(\xi_1,\xi_2)^T\in\mathcal{C}^2$, $\xi_i=\alpha_j x+\rho_j y+\omega_j t+\delta_j$, $j=1,2$. τ is a positive definite and real-valued symmetric 2×2 matrix which can taken of the form

$$\tau=\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad \tau_{11}>0, \quad \tau_{22}>0, \quad \tau_{11}\tau_{22}-\tau_{12}^2>0.$$

In order to get some sufficient conditions, such that the theta function (4.1) satisfies the bilinear equation (2.6), we substitute the function (4.1) into the left of Eq. (2.6) and obtain that

$$G(D_x,D_y,D_t)\vartheta(\xi_1,\xi_2,\tau)\vartheta(\xi_1,\xi_2,\tau)$$
$$=\sum_{m,n\in Z^2} G(2\pi i\langle n-m,\alpha\rangle,2\pi i\langle n-m,\rho\rangle,2\pi i$$
$$\times\langle n-m,\omega\rangle)e^{2\pi i\langle\xi,n+m\rangle-\pi(\langle\tau m,m\rangle+\langle\tau n,n\rangle)}$$
$$\overset{m=m'-n}{=}\sum_{m'\in Z^2}\sum_{n\in Z^2} G(2\pi i\langle 2n-m',\alpha\rangle,2\pi i\langle 2n-m',\rho\rangle,2\pi i$$
$$\times\langle 2n-m',\omega\rangle)\times\exp\{-\pi[\langle\tau(n-m'),n-m'\rangle$$
$$+\langle\tau n,n\rangle]\}\exp\{2\pi i\langle\xi,m'\rangle\}\equiv\sum_{m'\in Z^2}\bar{G}(m_1',m_2')e^{2\pi i\langle\xi,m'\rangle}.$$

In the last line we have introduced the notation $\bar{G}(m_1',m_2')$ for the coefficient of $e^{2\pi i\langle\xi,m'\rangle}$. For each fixed $l=1,2$, by shifting jth summation index as $n_j=n_j'+\delta_{j,l}$ with $\delta_{j,l}$ representing Kronecker's delta, we obtain that

$$\bar{G}(m_1',m_2')=\sum_{n\in Z^2} G(2\pi i\langle 2n-m',\alpha\rangle,2\pi i\langle 2n-m',\rho\rangle,2\pi i$$
$$\times\langle 2n-m',\omega\rangle)e^{-\pi[\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle]}$$
$$=\sum_{n\in Z^2} G\left(2\pi i\sum_{j=1}^{2}[2n_j'-(m_j'-2\delta_{jl})]\alpha_j,2\pi i\sum_{j=1}^{2}[2n_j'\right.$$
$$\left.-(m_j'-2\delta_{jl})]\rho_j,2\pi i\sum_{j=1}^{2}[2n_j'-(m_j'-2\delta_{jl})]\omega_j\right)$$
$$\times\exp\left\{-\pi\sum_{j,k=1}^{2}(n_j'+\delta_{jl})\tau_{jk}(n_k'+\delta_{kl})\right.$$

$$-\pi\sum_{j,k=1}^{2}[(m_j'-2\delta_{jl}-n_j')+\delta_{jl}]\tau_{jk}$$

$$\left.\times[(m_k'-2\delta_{kl}-n_k')+\delta_{kl}]\right\},$$

$$=\begin{cases}\bar{G}(m_1'-2,m_2')e^{-2\pi(\tau_{11}m_1'+\tau_{12}m_2')+2\pi\tau_{11}}, & l=1,\\ \bar{G}(m_1',m_2'-2)e^{-2\pi(\tau_{12}m_1'+\tau_{22}m_2')+2\pi\tau_{22}}, & l=2,\end{cases}$$

which implies that if the following equations are satisfied

$$\bar{G}(0,0)=\bar{G}(0,1)=\bar{G}(1,0)=\bar{G}(1,1)=0, \qquad (4.2)$$

then we have $\bar{G}(m_1',m_2')=0$ for all $m_1',m_2'\in Z$, and thus the function (4.1) is an exact solution of Eq. (2.6).

By introducing the notations as

$$M=(a_{jl}), \quad b=(b_1,b_2, \quad b_3,b_4)^T,$$

$$a_{j1}=-4\pi^2\sum_{n_1,n_2\in Z^2}\langle 2n-s^j,\alpha\rangle(2n_1-s_1^j)\varepsilon_j(n),$$

$$a_{j2}=-4\pi^2\sum_{n_1,n_2\in Z^2}\langle 2n-s^j,\alpha\rangle(2n_2-s_2^j)\varepsilon_j(n)$$

$$a_{j3}=4\pi^2\sum_{n_1,n_2\in Z^2}\langle 2n-s^j,\alpha\rangle\langle 2n-s^j,\rho\rangle\varepsilon_j(n),$$

$$a_{j4}=\sum_{n_1,n_2\in Z^2}\varepsilon_j(n),$$

$$b_j=-16\pi^4\sum_{n_1,n_2\in Z^2}\langle 2n-s^j,\alpha\rangle^3\langle 2n-s^j,\rho\rangle\varepsilon_j(n),$$

$$\varepsilon_j(n)=\lambda_1^{n_1^2+(n_1-s_1^j)^2}\lambda_2^{n_2^2+(n_2-s_2^j)^2}\lambda_3^{n_1n_2+(n_1-s_1^j)(n_2-s_2^j)},$$

$$\lambda_1=e^{-\pi\tau_{11}}, \quad \lambda_2=e^{-\pi\tau_{22}}, \quad \lambda_3=e^{-2\pi\tau_{12}},$$

$$s^j=(s_1^j,s_2^j), \quad j=1,2,3,4,$$

$$s^1=(0,0), \quad s^2=(1,0), \quad s^3=(0,1), \quad s^4=(1,1),$$

Eq. (4.2) can be written as a linear system

$$M(\omega_1,\omega_2,u_0,c)^T=b. \qquad (4.3)$$

Hence, we get an exact two-periodic wave solution to Eq. (1.1)

$$u=u_0+\frac{3}{2}\partial_x^2\ln\vartheta(\xi_1,\xi_2,\tau), \qquad (4.4)$$

with $\vartheta(\xi_1,\xi_2)$ and ω_1,ω_2,u_0,c given by Eqs. (4.1) and (4.3), respectively, while other parameters $\alpha_1,\alpha_2,\rho_1,\rho_2,\tau_{11},\tau_{22},\tau_{12}$ are free. The two-periodic wave is specified by six of the parameters $\alpha_1,\alpha_2,\rho_1,\rho_2,\tau_{11}$, and τ_{22}.

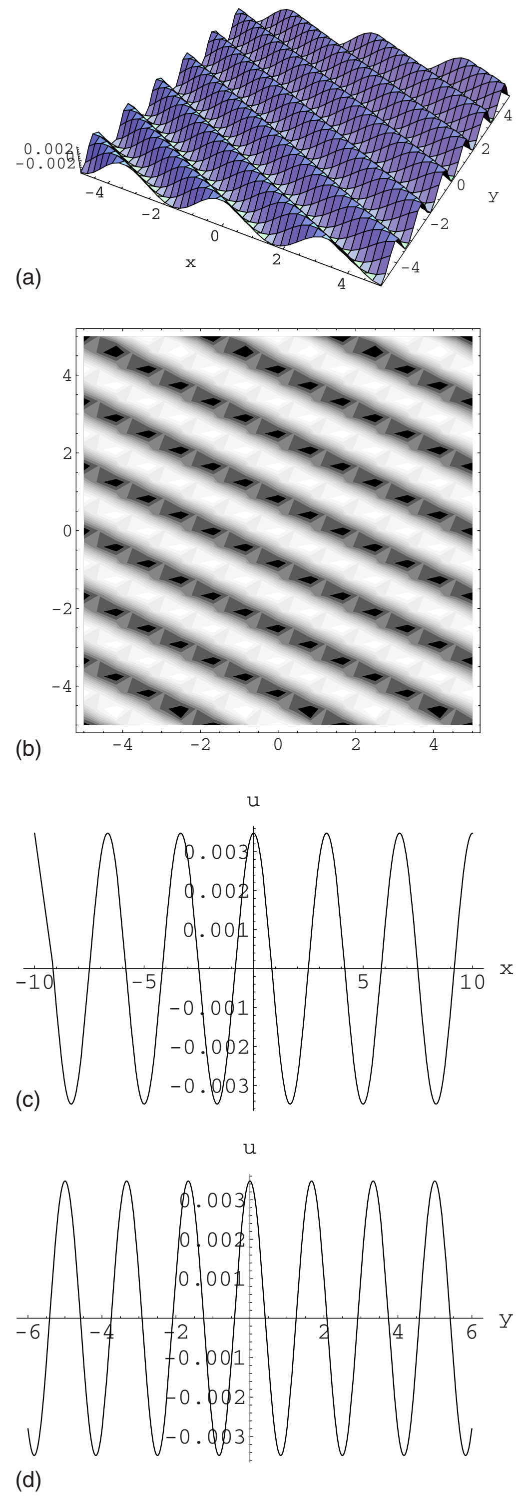

FIG. 2. (Color online) A degenerate two-periodic wave to (2+1)-dimensional Bogoyavlenskii's breaking soliton equation with parameters $\frac{\alpha_2}{\alpha_1}=\frac{\rho_2}{\rho_1}$ and $\alpha_1=0.01$, $\alpha_2=0.3$, $\tau_{11}=0.2$, $\tau_{12}=0.3$, $\tau_{22}=1$, $\rho_1=0.1$, $\rho_2=3$. This figure shows that the degenerate two-periodic wave is almost one dimensional. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs. (c) Wave propagation pattern of the wave along the x axis. (d) Wave propagation pattern of wave along the y axis.

B. Feature and asymptotic property of two-periodic waves

The two-periodic wave (4.4) has a simple characterization. (i) It is real valued and bounded for all complex variables (x,y,t). (ii) It is a direct generalization of one-periodic waves, its surface pattern is two dimensional, i.e., there are two phase variables ξ_1 and ξ_2. It has two independent spatial periods in two independent horizontal directions. The two-periodic wave may be considered to represent periodic waves in shallow water without the assumption of one dimensionality. (iii) It has $2N$ fundamental periods $\{e_j, j=1,\ldots,N\}$ and $\{i\tau_j, j=1,\ldots,N\}$ in (ξ_1,ξ_2). Its velocity of propagation is given by

$$\frac{dx}{dt}=\frac{\omega_2\alpha_1-\omega_1\alpha_2}{\alpha_1\rho_2-\alpha_2\rho_1}, \quad \frac{dy}{dt}=\frac{\omega_1\rho_2-\omega_2\rho_1}{\alpha_1\rho_2-\alpha_2\rho_1}.$$

(iv) If parameters satisfy a ratio relation

$$\frac{\alpha_2}{\alpha_1}=\frac{\rho_2}{\rho_1}=k \quad (k \text{ is a constant}),$$

then

$$\omega_2 \sim k\omega_1, \quad \xi_2 \sim k\xi_1, \quad \vartheta(\xi_1,\xi_2) \sim \vartheta(\xi_1,k\xi_1).$$

Therefore the two-periodic wave is actually one dimensional, and it degenerates to one-periodic wave (see Fig. 2). (v) If parameters do not satisfy a ratio relation, that is,

$$\frac{\alpha_2}{\alpha_1} \neq \frac{\rho_2}{\rho_1},$$

then for any time t, phase variables $\xi_1=\text{const}$ and $\xi_2=\text{const}$ intersect at a unique point. As the time t changes, this point moves in the (x,y) plane with a constant speed. In this case, the two-periodic solution is genuinely two dimensional, and it is spatially periodic in two independent directions in the (x,y) plane. Every two-periodic wave as in Fig. 3 is spatially periodic in two directions, but it need not be periodic in either the x or y directions. The basic cell of the pattern seems like a hexagon, but need not regular: six steep wave crests form the edges of each hexagon. The six crests surrounding a trough can be identified in pairs: opposite crests are parallel and have equal amplitudes as well as lengths along the crests. (vi) In a subcase of the above $\tau_{11}=\tau_{22}$, $\alpha_1=\alpha_2$, $\rho_1=-\rho_2$, the two-periodic solution has only three independent parameters $(\tau_{11},\alpha_1,\rho_1)$, and it is called a symmetric solution [33]. This solution is periodic both in x and y directions and propagate purely in the x direction. An example is shown in Fig. 4. It is seen that the cell of its pattern is a regular hexagon from the contour plot [see Fig. 4(b)].

At last, we consider the asymptotic properties of the two-periodic solution (4.4). In a similar way to theorem 1, we can establish the relation between the two-periodic solution (4.4) and the two-soliton solution (2.4) as follows.

Theorem 2. Assume that $(\omega_1,\omega_2,u_0,c)^T$ is a solution of the system (4.3), and for the two-periodic wave solution (4.4), we take

$$\alpha_j=\frac{\mu_j}{2\pi i}, \quad \rho_j=\frac{\nu_j}{2\pi i}, \quad \delta_j=\frac{\gamma_j+\pi\tau_{jj}}{2\pi i},$$

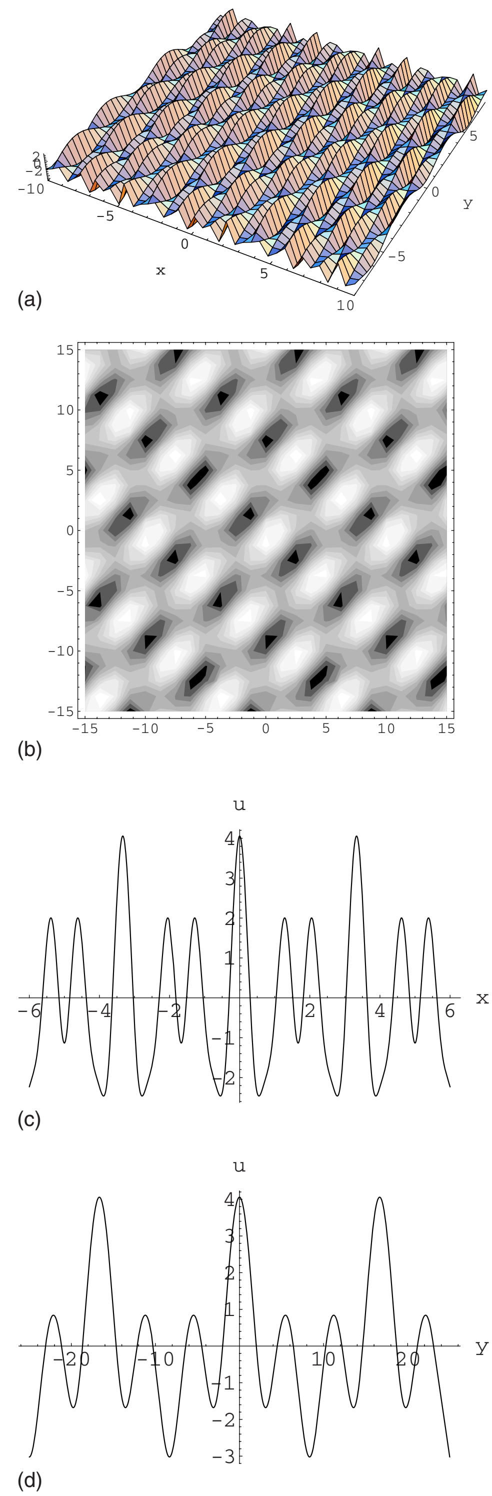

FIG. 3. (Color online) A asymmetric two-periodic wave for (2+1)-dimensional Bogoyavlenskii's breaking soliton equation with parameters $\alpha_1=0.6$, $\alpha_2=0.9$, $\tau_{11}=1$, $\tau_{12}=0.2$, $\tau_{22}=1.2$, $\rho_1=0.1$, $\rho_2=-0.2$. This figure shows that every two-periodic wave is spatially periodic in two directions, but it need not be periodic in either the x or y directions. (a) Perspective view of the wave. (b) Overhead view of the wave, with the contour plot shown. The bright hexagons are crests and the dark hexagons are troughs. (c) Wave propagation pattern of the wave along the x axis. (d) Wave propagation pattern of wave along the y axis.

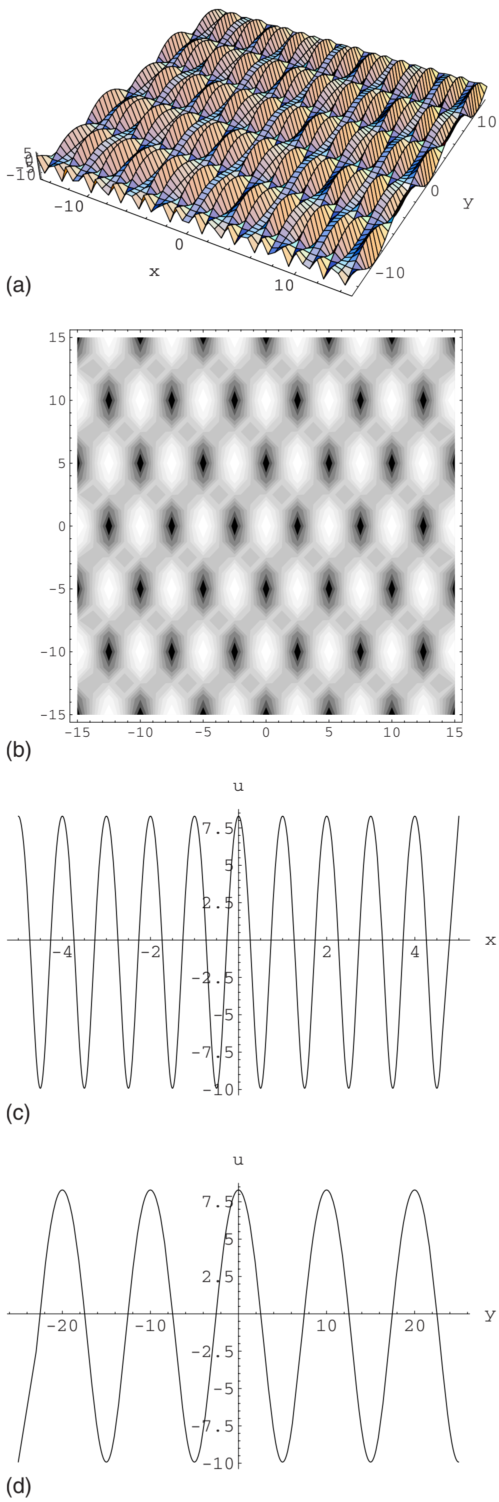

FIG. 4. (Color online) A symmetric two-periodic wave for (2+1)-dimensional Bogoyavlenskii's breaking soliton equation with parameters $\alpha_1=1$, $\alpha_2=1$, $\tau_{11}=1$, $\tau_{12}=0.2$, $\tau_{22}=1$, $\rho_1=0.1$, $\rho_2=-0.1$. This figure shows that the symmetric two-periodic wave is periodic in both x and y directions and propagate purely in the x direction. (a) Perspective view of the wave. (b) Overhead view of the wave with a contour plot shown. The bright hexagons are crests and the dark hexagons are troughs. (c) Wave propagation pattern of the wave along the x axis. (d) Wave propagation pattern of wave along the y axis.

$$\tau_{12}=\frac{A_{12}}{2\pi i}, \quad j=1,2, \tag{4.5}$$

with μ_j, ν_j, γ_j, $j=1,2$, and A_{12} as those given in Eq. (2.4). Then we have the following asymptotic relations:

$$u_0 \to 0, \quad c \to 0, \quad \xi_j \to \frac{\eta_j+\pi\tau_{jj}}{2\pi i}, \quad j=1,2,$$

$$\vartheta(\xi_1,\xi_2,\tau) \to 1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}},$$

$$\text{as } \lambda_1,\lambda_2 \to 0. \tag{4.6}$$

So the two-periodic solution (4.4) just tends to the two-soliton solution (2.4) under a certain limit, namely,

$$u \to u_2 \quad \text{as } \lambda_1,\lambda_2 \to 0.$$

Proof. We expand periodic wave function $\vartheta(\xi_1,\xi_2)$ in the following form:

$$\vartheta(\xi_1,\xi_2,\tau)=1+(e^{2\pi i\xi_1}+e^{-2\pi i\xi_1})e^{-\pi\tau_{11}} + (e^{2\pi i\xi_2}+e^{-2\pi i\xi_2})e^{-\pi\tau_{22}}+(e^{2\pi i(\xi_1+\xi_2)} + e^{-2\pi i(\xi_1+\xi_2)})e^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})}+\cdots .$$

Further by using Eq. (4.5) and making a transformation $\hat{\omega}_j=2\pi i\omega_j$, $j=1,2$, we get

$$\vartheta(\xi_1,\xi_2,\tau)=1+e^{\hat{\xi}_1}+e^{\hat{\xi}_2}+e^{\hat{\xi}_1+\hat{\xi}_2-2\pi\tau_{12}}+\lambda_1^2e^{-\hat{\xi}_1}+\lambda_2^2e^{-\hat{\xi}_2} +\lambda_1^2\lambda_2^2e^{-\hat{\xi}_1-\hat{\xi}_2-2\pi\tau_{12}}+\cdots \to 1+e^{\hat{\xi}_1}+e^{\hat{\xi}_2} +e^{\hat{\xi}_1+\hat{\xi}_2+A_{12}}, \quad \text{as } \lambda_1,\lambda_2\to 0,$$

where $\hat{\xi}_j=\mu_j x+\nu_j y+\hat{\omega}_j t+\gamma_j$, $j=1,2$.

It remains to prove that

$$c \to 0, \quad \hat{\omega}_j \to -\mu_j^2\nu_j, \quad \hat{\xi}_j \to \eta_j, \quad j=1,2,$$

$$\text{as } \lambda_1,\lambda_2 \to 0. \tag{4.7}$$

As in Eq. (3.15), we can expand each function in $\{a_{ij},b_j,j=1,2,3,4\}$ into a series with λ_1, λ_2. It is slightly more tedious than that Eq. (3.15), but this process is easily carried out by using symbolic computation software MATHEMATICA or MAPLE. Actually, we only need to make the first order expansions of matrix M and vector b with λ_1, λ_2 to show the asymptotic relations (4.7). Here we consider their second order expansions to see deeper relations among parameters for the two-periodic solution (4.4) and the two-soliton solution (2.4). The expansions for the matrix M and the vector b are given by

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -8\pi^2\alpha_1 & 0 & -8\pi^2\alpha_1\rho_1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\lambda_1 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -8\pi^2\alpha_2 & -8\pi^2\alpha_2\rho_2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\lambda_2$$
$$+ \begin{pmatrix} -32\pi^2\alpha_1 & 0 & -32\pi^2\alpha_1\rho_1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\lambda_1^2 + \begin{pmatrix} 0 & -32\pi^2\alpha_2 & -32\pi^2\alpha_2\rho_2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\lambda_2^2$$
$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8\pi^2(\alpha_2-\alpha_1) & 8\pi^2(\alpha_1-\alpha_2)-8\pi^2(\alpha_1+\alpha_2)\lambda_3 & 8\pi^2(\alpha_2-\alpha_1)(\rho_1-\rho_2) & 2 \end{pmatrix}\lambda_1\lambda_2 + o(\lambda_1^k\lambda_2^j), \quad k+j \geq 2 \tag{4.8}$$

and

$$b = \begin{pmatrix} 0 \\ -32\pi^4\alpha_1^3\rho_1 \\ 0 \\ 0 \end{pmatrix}\lambda_1 + \begin{pmatrix} 0 \\ 0 \\ -32\pi^4\alpha_2^3\rho_2 \\ 0 \end{pmatrix}\lambda_2 + \begin{pmatrix} -512\pi^4\alpha_1^3\rho_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\lambda_1^2 + \begin{pmatrix} 0 \\ -512\pi^4\alpha_2^3\rho_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}\lambda_2^2$$
$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -32\pi^4(\alpha_1+\alpha_2)^3(\rho_1+\rho_2)\lambda_3 - 32\pi^4(\alpha_1-\alpha_2)^3(\rho_1-\rho_2) \end{pmatrix}\lambda_1\lambda_2 + o(\lambda_1^k\lambda_2^j), \quad k+j \geq 2, \tag{4.9}$$

where $o(\lambda_1^k\lambda_2^j)$ denote higher infinitesimal than $\lambda_1^k\lambda_2^j$, $k+j \geq 2$.

We also assume the solution of the system (4.3) in the following form:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ u_0 \\ c \end{pmatrix} = \begin{pmatrix} \omega_1^{(0)} \\ \omega_2^{(0)} \\ u_0^{(0)} \\ c^{(0)} \end{pmatrix} + \begin{pmatrix} \omega_1^{(1)} \\ \omega_2^{(1)} \\ u_0^{(1)} \\ c^{(1)} \end{pmatrix}\lambda_1 + \begin{pmatrix} \omega_1^{(2)} \\ \omega_2^{(2)} \\ u_0^{(2)} \\ c^{(2)} \end{pmatrix}\lambda_2 + \begin{pmatrix} \omega_1^{(11)} \\ \omega_2^{(11)} \\ u_0^{(11)} \\ c^{(11)} \end{pmatrix}\lambda_1^2$$
$$+ \begin{pmatrix} \omega_1^{(22)} \\ \omega_2^{(22)} \\ u_0^{(22)} \\ c^{(22)} \end{pmatrix}\lambda_2^2 + \begin{pmatrix} \omega_1^{(12)} \\ \omega_2^{(12)} \\ u_0^{(12)} \\ c^{(12)} \end{pmatrix}\lambda_1\lambda_2 + o(\lambda_1^k\lambda_2^j), k+j \geq 2. \tag{4.10}$$

Substituting Eqs. (4.8)–(4.10) into Eq. (4.3) and comparing the same order of λ_1, λ_2, we obtain the following relations:

$$c^{(0)} = c^{(1)} = c^{(2)} = c^{(12)} = 0,$$

$$\omega_1^{(0)} + \rho_1 u_0^{(0)} = 4\pi^2\alpha_1^2\rho_1, \omega_2^{(0)} + \rho_2 u_0^{(0)} = 4\pi^2\alpha_2^2\rho_2,$$

$$\omega_1^{(1)} + \rho_1 u_0^{(1)} = 0, \quad \omega_2^{(1)} + \rho_2 u_0^{(1)} = 0,$$

$$c^{(11)} - 32\pi^2\alpha_1\omega_1^{(0)} - 32\pi^2\alpha_1\rho_1 u_0^{(0)} = -512\pi^4\alpha_1^3\rho_1,$$

$$c^{(22)} - 32\pi^2\alpha_2\omega_2^{(0)} - 32\pi^2\alpha_2\rho_2 u_0^{(0)} = -512\pi^4\alpha_2^3\rho_2.$$

To make the relations (4.7) hold, we choose $u_0^{(0)}=0$, and thus

$$u_0 = o(\lambda_1,\lambda_2) \to 0,$$

$$c = -384\pi^4\alpha_1^3\rho_1\lambda_1^2 - 384\pi^4\alpha_2^3\rho_2\lambda_2^2 + o(\lambda_1^2,\lambda_2^2) \to 0,$$

$$\omega_1 = 4\pi^2\alpha_2^2\rho_1 + o(\lambda_1,\lambda_2) \to 4\pi^2\alpha_1^2\rho_1,$$

$$\omega_2 = 4\pi^2\alpha_2^2\rho_2 + o(\lambda_1,\lambda_2) \to 4\pi^2\alpha_2^2\rho_2, \quad \text{as } \lambda_1,\lambda_2 \to 0,$$

which implies Eq. (4.7). Therefore we conclude that the two-periodic solution (4.4) tends to the two-soliton solution (2.4) as $\lambda_1,\lambda_2 \to 0$. ■

In this paper, we consider one- and two-periodic wave solutions of the Bogoyavlenskii's breaking soliton equation (1.1), which belong to the cases when $N=1$ and $N=2$. The results can be extended to the case when $N>2$, but there are still certain numerical difficulties in the calculation, which will be considered in our future work.

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